REFERENCES

1. T. L. Bergman, F. P. Incropera and R. Viskanta, A multilayer model for mixing layer development in a double-diffusive thermohaline system heated from below, *Int. J. Heat Mass Transfer* **25**, 1411–1418 (1982).
2. T. L. Bergman, F. P. Incropera and R. Viskanta, A differential model for salt-stratified, double-diffusive systems heated from below, *Int. J. Heat Mass Transfer* **28**, 779–788 (1985).
3. C. E. Nielsen, Salt transport and gradient maintenance in solar ponds, *Proc. AS-ISES*, Houston, pp. 179–184 (1982).
4. United States Office of Saline Water, Technical Data Book (1964).
5. J. S. Turner, The behaviour of a stable salinity gradient heated from below, *J. Fluid Mech.* **33**, 183–200 (1968).
6. H. E. Huppert and P. F. Linden, On heating a stable salinity gradient from below, *J. Fluid Mech.* **95**, 431–464 (1979).

Int. J. Heat Mass Transfer. Vol. 28, No. 8, pp. 1621–1624, 1985
Printed in Great Britain

0017-9310/85 \$3.00 + 0.00
Pergamon Press Ltd.

Effects of magnetic field and non-uniform temperature gradient on Marangoni convection

N. RUDRAIAH, V. RAMACHANDRAMURTHY

UGC-DSA Centre in Fluid Mechanics, Department of Mathematics, Central College, Bangalore University, Bangalore-560001, India

and

O. P. CHANDNA

Department of Mathematics, University of Windsor, Windsor, Canada N9B 3P4

(*Received December* 1983)

NOMENCLATURE

a	resultant horizontal wave number, $(l^2+m^2)^{1/2}$
d	thickness of the fluid layer
D	d/dz
$f(z)$	a non-dimensional temperature gradient, $-\dfrac{d}{\Delta T}\dfrac{dT_0}{dz}$
$\mathbf{H}$	magnetic field
$\hat{\mathbf{k}}$	unit vector in the z-direction
l, m	horizontal wave numbers in the directions of x- and y-axes
M	Marangoni number, $\dfrac{\sigma_T \Delta T d}{\rho\delta\kappa}$
M_c	critical Marangoni number
Pr	magnetic Prandtl number, δ/δ_m
Pr	Prandtl number, δ/κ
$\mathbf{q}$	velocity, (u, v, w)
Q	Chandrasekhar number, $\dfrac{\mu H_0^2 d^2}{\rho\delta\delta_m}$
T	temperature
T_0	basic temperature
ΔT	temperature difference between two boundaries
t	time
W	z-component of velocity
(x, y, z)	Cartesian coordinates
∇^2	$\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$.

Greek symbols

α	a^2
κ	thermal diffusivity
μ	magnetic permeability
ν	kinematic viscosity
ν_m	magnetic viscosity
ρ	mass density
σ	electrical conductivity
σ_T	variation of surface tension with temperature
σ_0, σ_1	constants
ε	quasi-time-dependent thermal depth
δ	Dirac's delta-function.

INTRODUCTION

THE MECHANISM of controlling convection generated in a fluid either by the buoyancy force or by the surface tension variation with temperature or by both has recently assumed importance in material processing in space [1] because of its application to the possibility of producing various new materials. The range of possibilities extends from producing large crystals of uniform properties to manufacturing materials with unique properties. The reduced gravity (i.e. microgravity) environment provided during sustained space flight prevents convection generated by the buoyancy force called Rayleigh–Bénard convection. But the other type of convection generated by variation of surface tension with temperature [2, 3] called Marangoni convection [4], can still exist. Such convection may also influence the local material composition and the shape of the solid–liquid interface. This can result in materials with non-uniform properties and crystal defects. Forces like Lorentz force due to electromagnetic effects [5, 6], coriolis force due to rotation [7] and non-uniform temperature gradient due to transient heating or cooling [8] at the boundaries, which are ineffective at the terrestrial environment, become effective in the microgravity environment. Such forces may be used to suppress or augment Marangoni convection.

The effect of non-uniform temperature gradient on buoyancy driven convection is now well known [9–11]. Recently Rudraiah and Friedrich [7] and Rudraiah [8] have studied the effect of coriolis force on Marangoni convection and have shown that rotation suppreses convection. Nield [6] has examined the effect of magnetic field on convection driven by combined buoyancy and surface tension forces with uniform basic temperature gradient. The combined effect of non-uniform temperature gradient and magnetic eifld on Marangoni convection has not been given much attention, although they have significant effect on the fluid aboard the space craft. Therefore, the object of this paper is to show analytically using the Galerkin technique that a suitable non-uniform temperature gradient and magnetic field suppress Marangoni convection. Comparison of analytical results with the numerical results of Lebon and Cloot [12] in the absence of magnetic field and with those of Nield [6] in the presence of magnetic field with uniform temperature gradient reveals that a single-term Galerkin expansion procedure used here gives reasonable results with minimum mathematics.

2. FORMULATION OF THE PROBLEM

We consider an infinite horizontal layer of an electrically conducting liquid permeated by a uniform vertical magnetic field. The mean depth of the liquid layer is d. It is bounded below by a rigid, electrically- and thermally-perfect conducting wall and bounded above by a free surface. This free surface is adjacent to an electrically non-conducting medium and subject to a constant heat flux (i.e. adiabatic). We assume a temperature drop ΔT acting between the upper and lower boundaries. The interface has a surface tension σ which, following Pearson [4], can be assumed to vary linearly with temperature according to the formula

$$\sigma = \sigma_0 - \sigma_1 \Delta T. \tag{1}$$

We use the Cartesian coordinate system (x, y, z) with the origin at the bottom of the boundary, $0x$ parallel to the boundaries and $0y$ and $0z$ normal to them. Let the applied uniform magnetic field H_0 acts in the z-direction. The governing equations for this configuration are well known (see Chandrasekhar [5]).

In the quiescent state the velocity $\mathbf{q}$, the temperature T and the magnetic field $\mathbf{H}$ have the following solutions

$$\mathbf{q} = 0, \quad \mathbf{H} = H_0\hat{\mathbf{k}}, \quad -\frac{d}{\Delta T}\frac{\mathrm{d}T_0}{\mathrm{d}z} = f(z) \tag{2}$$

where $\hat{\mathbf{k}}$ is the unit vector in the z-direction and $f(z)$ is a non-dimensional temperature gradient satisfying the condition

$$\int_0^1 f(z)\,\mathrm{d}z = 1. \tag{3}$$

Suppose that the initial state is slightly disturbed. The linearized equations of motion allow the solution of a disturbance in the form

$$\text{(some function of } z) \exp\{\mathrm{i}(lx+my)+\omega t\}$$

where l and m are the wave numbers and ω is the growth rate. We use this expression in the linearized version of basic equations and eliminate x and y components of the velocity and the induced magnetic field. After some manipulations and making the resulting equations dimensionless using d, d^2/v, v/d, $\delta/a\sqrt{\Delta T\rho\delta/d\sigma_T\kappa}$ and H_0 as length, time, velocity, temperature and induced magnetic field scales, respectively, we obtain

$$(D^2-a^2-\omega)(D^2-a^2)W + GD(D^2-a^2)H = 0 \tag{4}$$

$$(D^2-a^2-\omega P_m)H + P_m DW = 0 \tag{5}$$

$$(D^2-a^2-\omega P_r)T + aM^{1/2}f(z)W = 0 \tag{6}$$

where $D = \mathrm{d}/\mathrm{d}z$, $a^2 = l^2+m^2$, $G = \mu H_0^2 d^2/\rho v^2$ is the magnetic parameter analogous to Grashof number, $Pr = v/\kappa$ is the Prandtl number, $Pr_m = v/v_m$ is the magnetic Prandtl number, $M = \sigma_T d\Delta T/\rho v\kappa$ is the Marangoni number, σ_T is the variation of surface tension with temperature, v is the kinematic viscosity, v_m is the magnetic viscosity, κ is the thermal diffusivity, and $W(z)$, $H(z)$ and $T(z)$ are, respectively, the amplitudes of the z-component of the velocity, magnetic field, and temperature distribution. When the principle of exchange of stabilities (i.e. $\omega = 0$) holds equations (4)–(6) are considerably simplified. From these equations we may eliminate H to obtain equations for W and T to be

$$(D^2-a^2)^2 W = QD^2W \tag{7}$$

$$(D^2-a^2)T + aM^{1/2}f(z)W = 0 \tag{8}$$

where $Q = GPr_m = \mu H_0^2 d^2/\rho v v_m$ is the Chandrasekhar number. The corresponding boundary conditions are [7]

$$W = DW = T = 0 \quad \text{at} \quad z = 0. \tag{9}$$

$$W = D^2W + aM^{1/2}, \quad T = DT = 0 \quad \text{at} \quad z = 1. \tag{10}$$

3. CONDITION FOR THE ONSET OF MARANGONI CONVECTION

Multiplication of equation (7) by W, of equation (8) by T, integration of the resulting equations by parts with respect to z from 0 to 1, using the boundary conditions (9) and (10) and using $W = AW_1$, $T = BT_1$ in which A and B are constants and W_1 and T_1 are the trial functions, yields the following eigenvalue equation

$$M = [-\langle (DT)^2 + a^2T^2\rangle\langle (D^2W)^2 + (2a^2+Q) \times (DW)^2 + a^4W^2\rangle]/[a^2DW(1)T(1)\langle f(z)WT\rangle] \tag{11}$$

where the angle bracket $\langle\ldots\rangle$ denotes the integration with respect to z from 0 to 1. We select the trial functions

$$W = z^2(1-z^2), \quad T = z(1-z/2) \tag{12}$$

such that they satisfy all the boundary conditions except the one given by $D^2W + aM^{1/2}T = 0$ at $z = 1$, but the residual from this is included in a residual from the differential equations. Substituting equations (12) into (11) and performing the integration,

$$M = [16\alpha^3 + 568\alpha^2 + (11904 + 264Q)\alpha + 660Q + 26460] \Big/ \left[4725\alpha\left\langle f(z)\left(z^3 - \frac{z^4}{2} - z^5 + \frac{z^6}{2}\right)\right\rangle\right] \tag{13}$$

where $\alpha = a^2$.

For any given $f(z)$, M attains its minimum value at $\alpha_c = a_c^2$, α_c being the root of the cubic equation

$$\alpha^3 + 17.75\alpha^2 - (20.625Q + 826.875) = 0. \tag{14}$$

The variation of α_c with Q is computed for different values of Q and the results are given in Table 1. From this it is clear that, as in magnetoconvection driven by the buoyancy force [5], the critical wave number increases with increasing Q and hence the effect of magnetic field is to contract the Marangoni cells. When the layer of conducting liquid is heated from below, the non-uniform temperature gradient $f(z)$ is not only non-negative but also decreases monotonically. Thus, we are interested in the temperature profile which gives the maximum M_c subject to $f(z) \geqslant 0$.

Therefore, as in the case of rotation [7], we consider the following profiles:

(i) $f(z) = 1$ corresponds to uniform temperature gradient;

Table 1. Critical Marangoni number for different values of Q

Q	α_c	$(M_c)_1$	$(M_c)_2$	$(M_c)_3$	$(M_c)_4$	$(M_c)_5$
0	2.43133	78.44205	75.95960	47.71781	37.19355	116.39788
10^{-1}	2.43271	78.58722	76.10017	47.80612	37.26238	116.61329
10^0	2.44470	79.89147	77.36317	48.59952	37.88081	118.54863
10^1	2.55419	92.74618	89.81105	56.41928	43.97591	137.62337
10^2	3.18943	211.89126	128.89751	128.89751	100.46893	314.41928
10^3	4.79127	1264.8422	1224.8138	769.42775	599.72905	1876.8626
10^4	7.33472	11021.928	10673.118	6704.8501	5526.083	16355.118

$$\text{(ii)}\ f(z) = \begin{cases} 1/\varepsilon & \text{for } 0 \leqslant z < \varepsilon \\ 0 & \text{for } \varepsilon < z \leqslant 1 \end{cases}$$

corresponds to a piecewise linear profile arising from sudden heating from below where ε is a time dependent thermal depth parameter;

$$\text{(iii)}\ f(z) = \begin{cases} 0 & \text{for } 0 \leqslant z < 1-\varepsilon \\ 1/\varepsilon & \text{for } 1-\varepsilon < z \leqslant 1 \end{cases}$$

corresponds to a piecewise linear profile for sudden cooling from above;

(iv) $f(z) = \delta(z-\varepsilon)$ corresponds to a superposition of two layers at different temperature in which the basic temperature drops suddenly by an amount ΔT at $z = \varepsilon$ but is otherwise uniform, where δ is the Dirac delta-function; and

(v) $f(z) = 2(1-z)$, corresponds to an inverted parabolic temperature profile generated in a layer of conducting fluid through Joule heating with an alternating current. In these cases the Marangoni numbers, denoted respectively by M_i ($i = 1$–5) and the corresponding critical values $(M_c)_i$ are computed from equation (13) for different values of Q. The results are given in Table 1, and are discussed in the next section.

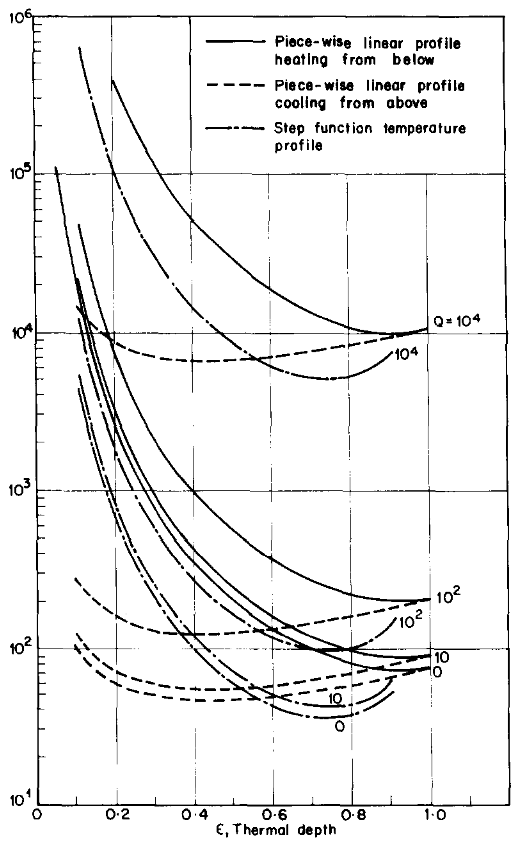

FIG. 1. Critical Marangoni number as a function of thermal depth for different values of Q.

4. DISCUSSION AND CONCLUSIONS

The purpose of this paper has been to study the effects of a non-uniform temperature gradient and a uniform transverse magnetic field on the linear stability of a horizontal layer of a conducting liquid at rest with the object of knowing which temperature profile gives the maximum critical Marangoni number. The single-term Galerkin procedure provides a quick method for establishing the above object and the following conclusions have been drawn:

(1) A comparison of the critical Marangoni number in Table 1 shows that the system is more unstable in the case of the superposed two-fluid model because the temperature jump occurs nearer the less restrictive free surface. This comment also applies to the case when the layer is cooled from above where the destabilizing force is applied nearer the less restrictive free surface.

(2) In all the cases we have seen that the critical Marangoni number increases with increasing Chandrasekhar number Q. Thus, the magnetic field inhibits the onset of Marangoni convection.

(3) The critical Marangoni numbers computed for different values of ε and Q are shown in Fig. 1. It is seen that as ε increases from 0 to 1, M_c decreases to a minimum and then increases again.

(4) The magnetic field and inverted parabolic basic temperature profile increases M_c considerably. Hence they make the system more stable than in all other cases.

(5) From Table 1 it is clear that the critical wave number increases with increasing Q having the asymptotic behaviour

$$a_c \to (1.6560)^{1/6} \quad \text{as} \quad Q \to \infty. \tag{15}$$

In other words, the critical wave number depends crucially on Q but is independent of the nature of the basic temperature profile. The asymptotic behaviour of M_c, however, depends on the basic temperature profile and on Q as shown in Table 2. The asymptotic value of M_c shown in Table 2 is proportional to Q and is analogous to the one given in [5] for Rayleigh–

Table 2. Asymptotic behaviour of M_c as $Q \to \infty$

M_c for $Q \to \infty$	ε_c
$(M_c)_1 \to 1.02Q$	1
$(M_c)_2 \to 0.988Q$	0.932
$(M_c)_3 \to 0.621Q$	0.428
$(M_c)_4 \to 0.484Q$	0.739
$(M_c)_5 \to 1.51Q$	—

Table 3. Comparison with hydrodynamic ($Q = 0$) results

Temperature gradient	Present analysis			Lebon and Cloot [12]		
	M_c	a_c	ε_c	M_c	a_c	ε_c
Uniform	78.44	2.43	—	79.61	1.99	—
Heated from below	75.96	2.43	0.932	78.1	2.025	0.96
Cooled from above	47.72	2.43	0.428	42.62	2.25	0.675
Superposed two-fluid layer	37.19	2.43	0.739	34.3	2.305	0.815

Bénard magnetoconvection with uniform temperature gradient. From Tables 1 and 2, it is clear that

$$(M_c)_4 < (M_c)_3 < (M_c)_2 < (M_c)_1 < (M_c)_5. \qquad (16)$$

(6) The results obtained from the present analytical analysis are compared with the numerical results of Lebon and Cloot [12] for $Q = 0$ in Table 3 and with the numerical results of Nield [6] for $Q \neq 0$ and $f(z) = 1$ in Table 4.

From Table 3 it is clear that our analytical results are in close agreement with the elaborate numerical exploration of the modified Tchebyschev polynomials given in [12]. From Table 4 it is clear that for small values of Q our results are in close agreement with the numberical results of Nield [6] obtained using the Fourier series method of Chandrasekhar [5]. The slight deviation for large values of Q is mainly due to the deviation in the critical wave number a_c. From Tables 3 and 4, we conclude that the single-term Galerkin procedure used here gives reasonable results with minimum of mathematics.

The above results are true only for infinitesimal disturbances. The work is in progress to include finite amplitude disturbances.

Table 4. Comparison with numerical results of Nield [6] for $f(z) = 1$

Q	Present analysis		Nield [6]	
	M_cM_c	a_ca_c	M_c	a_c
0	78.44	2.43	79.607	1.99
2.5	82.06	2.46	85.971	2.05
12.5	92.26	2.58	110.08	2.22
25	113.58	2.71	138.09	2.39
50	147.18	2.90	189.87	2.63
125	243.45	3.30	328.69	3.09
250	396.77	3.71	536.90	3.54
500	692.21	4.21	919.77	4.08
1000	1264.84	4.79	1632.50	4.76
2500	2932.74	5.69	3625.00	5.84
5000	5654.81	6.46	6673.00	6.87
10000	11021.93	7.33	12828.00	8.11

Acknowledgement—The work of NR and VR was supported by the UGC under the DSA programme, while the work of OPC was supported by the National Research Council for Canada.

REFERENCES

1. K. Oshima, Thermal and fluid mechanical problems in fluid mechanics, *Indian Acad. Sci.* 269–281 (1982).
2. J. Zierep and H. Oertel, Jr., *Convective Transport and Instability Phenomena.* G. Braun, Karlsruhe (1982).
3. S. Ostrach, The influence of convection in continuous-flow electrophoresis, ESA Special Publication 114, pp. 141–147 (1979).
4. J. R. A. Pearson, On convection cells induced by surface tension, *J. Fluid Mech.* **4,** 489–500 (1958).
5. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability.* Clarendon Press, Oxford (1961).
6. D. A. Nield, Surface tension and buoyancy effects in the cellular convection of an electrically conducting liquid in a magnetic field, *Z. angew. Math. Mech.* **17,** 131–139 (1966).
7. R. Friedrich and N. Rudraiah, Marangoni convection in a rotating fluid layer with non-uniform temperature gradient, *Int. J. Heat Mass Transfer* **27,** 443–449 (1984).
8. N. Rudraiah, The onset of transient Marangoni convection in a liquid layer subjected to rotation about a vertical axis, *Mater. Sci. Bull. Indian Acad. Sci.* **4,** 297–316 (1982).
9. D. A. Nield, The onset of transient convective instability, *J. Fluid Mech.* **71,** 441–454 (1975).
10. N. Rudraiah, B. Veerappa and S. Balachandra Rao, Effects of non-uniform thermal gradient and adiabatic boundaries on convection in porous media, *Trans. Am. Soc. mech. Engrs,* Series C, *J. Heat Transfer* **102,** 254–260 (1980).
11. N. Rudraiah, B. Veerappa and S. Balachandra Rao, Convection in a fluid-saturated porous layer with non-uniform temperature gradient, *Int. J. Heat Mass Transfer* **25,** 1147–1150 (1982).
12. G. Lebon and A. Cloot, Effects of non-uniform temperature gradients on Bénard–Marangoni's instability, *J. non-equilib. Thermodyn* **6,** 15–30 (1981).